The Introduction To Artificial Intelligence

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The Introduction to Artificial Intelligence

- Part I Brief Introduction to AI & Different AI tribes
- Part II Knowledge Representation & Reasoning
- Part III AI GAMES and Searching
- Part IV Model Evaluation and Selection
- Part V Machine Learning

Machine Learning

Supervised learning

Unsupervised learning

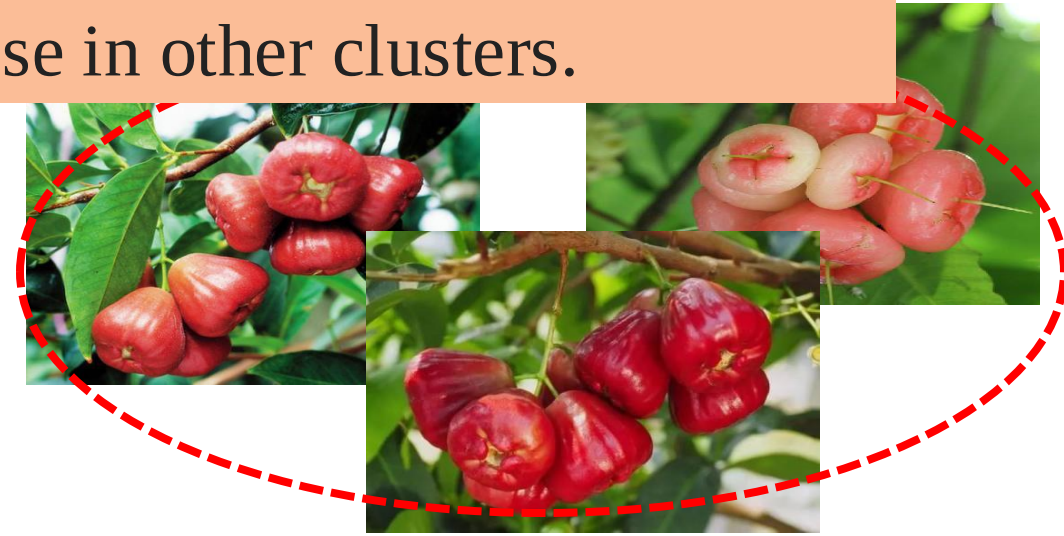
Reinforcement learning

Unsupervised learning

- Clustering
  - K-means method
  - Spectral clustering
- Representation learning

Clustering

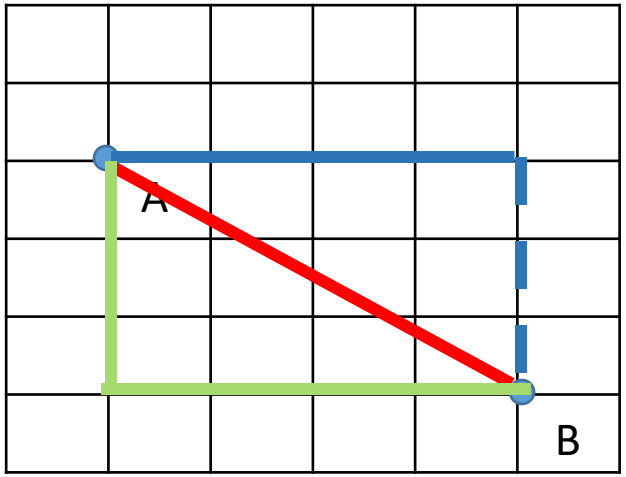
Clustering: Grouping a set of data in such a way that data in the same cluster are more similar to each other than to those in other clusters.



Clustering

Distance Metrics

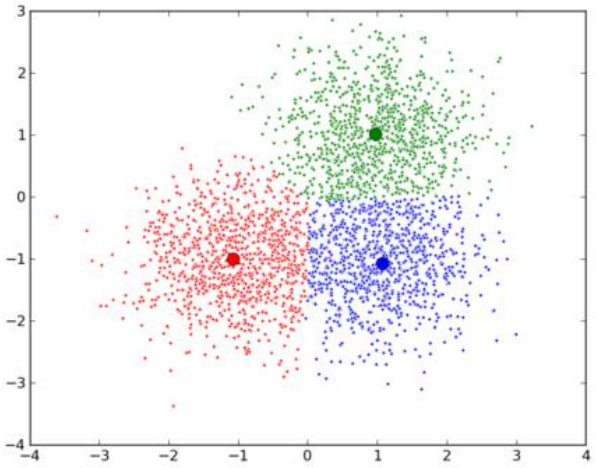
- Euclidean distance
- $d_e(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
- Sum of squared distance
- $d_q(x, y) = \sum_{i=1}^{n}(x_i - y_i)^2$
- Manhattan distance
- $d_m(x, y) = \sum_{i=1}^{n}|x_i - y_i|$
- Chebyshev distance
- $d_c(x, y) \max_{i=1,\cdots,n}|x_i - y_i|$



Clustering

K-means Clustering

- K-means clustering is a sort of clustering algorithm, and it is popular for cluster analysis in data mining.
- K-means clustering aims to partition N observations into K clusters in which each observation belongs to the cluster with the nearest mean, serving as a prototype of the cluster.

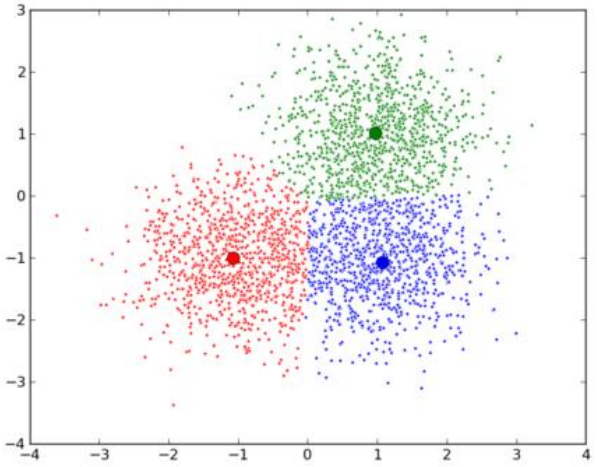


Clustering

K-means Clustering

- High intra-clustering similarity
- Low inter-clustering similarity.
- So,

$$object : \sum_{i=1}^{N} \min_{u_j \in C} \|x_i - u_j\|^2$$



Clustering

- K-means Clustering

e.g. k =2

First the k cluster midpoints $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k$ are randomly or manually initialized.

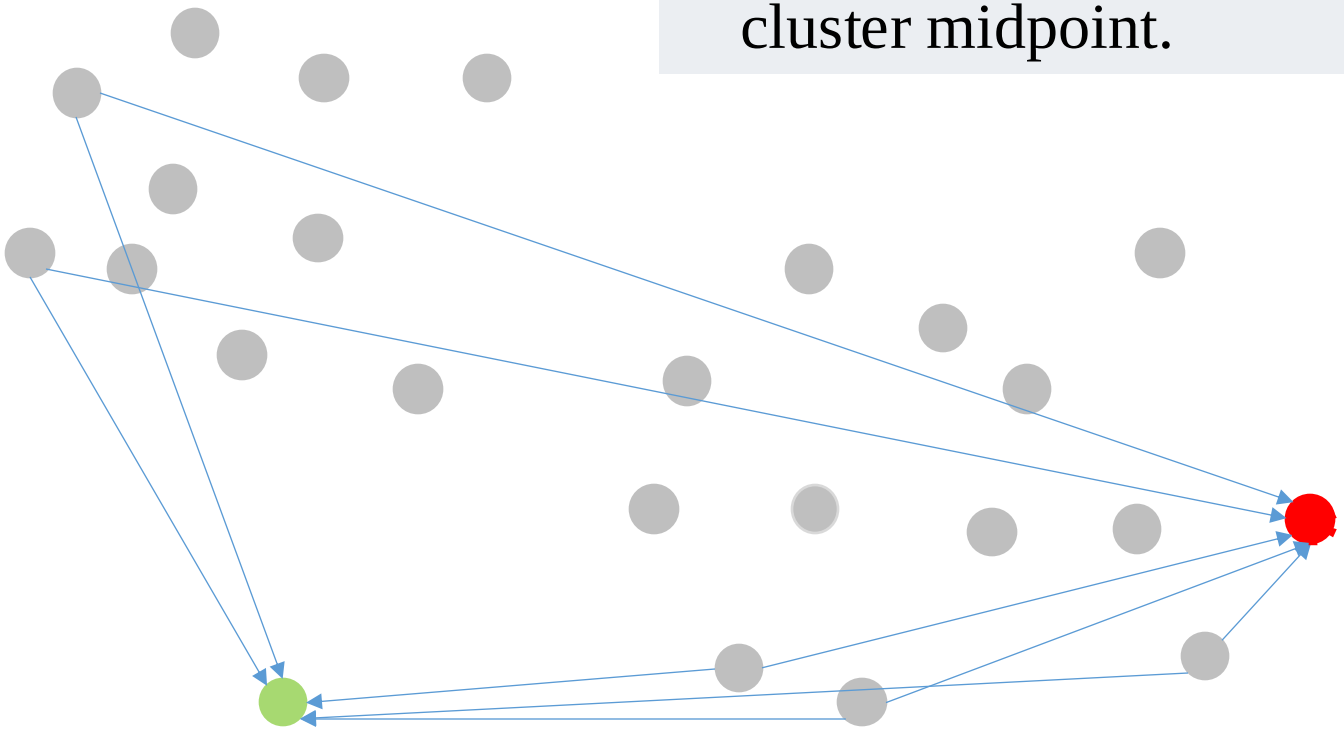
Clustering

- K-means Clustering

e.g. k =2

Then the following two steps are repeatedly carried out:

- Classify all data to their nearest cluster midpoint.



Clustering

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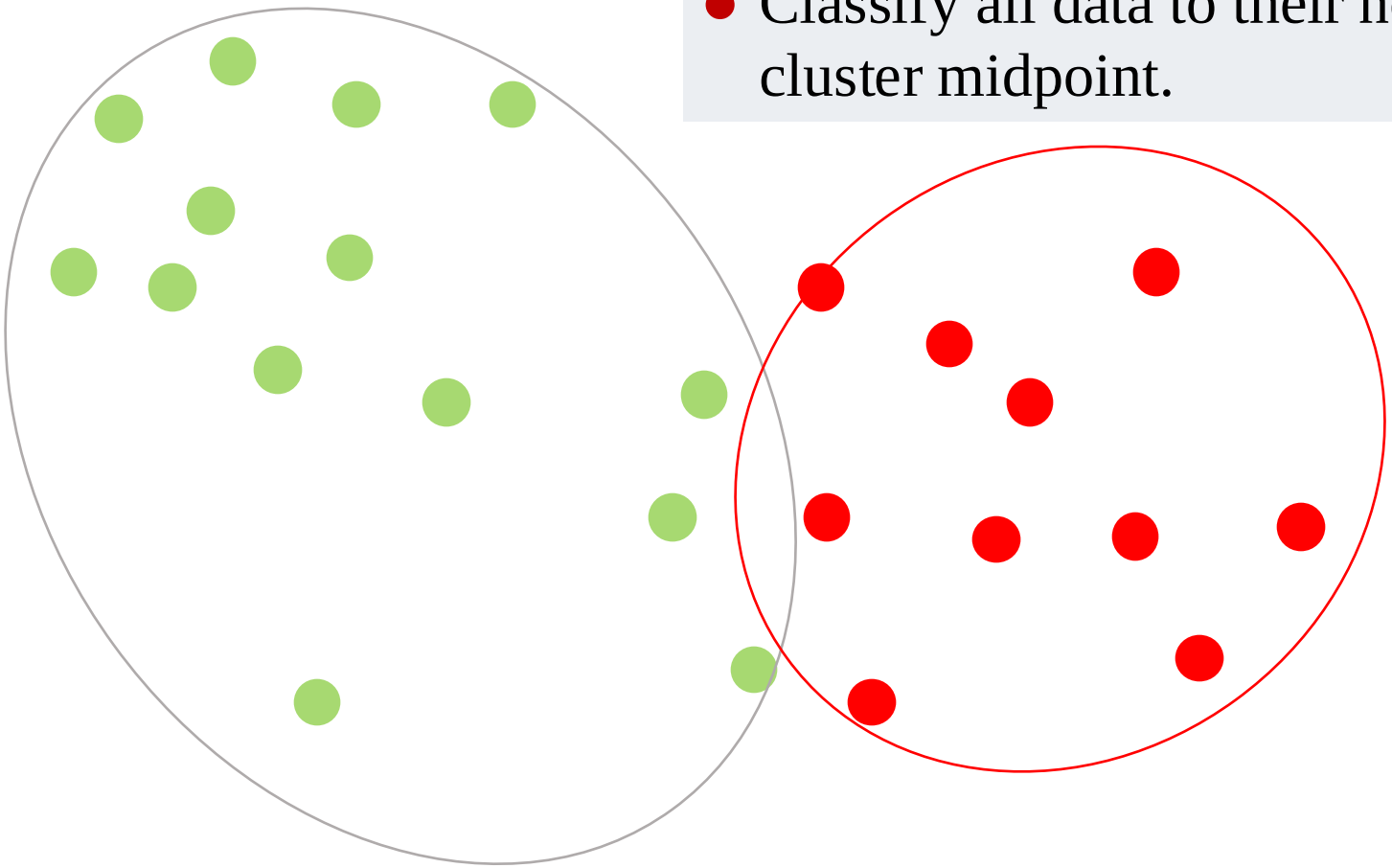


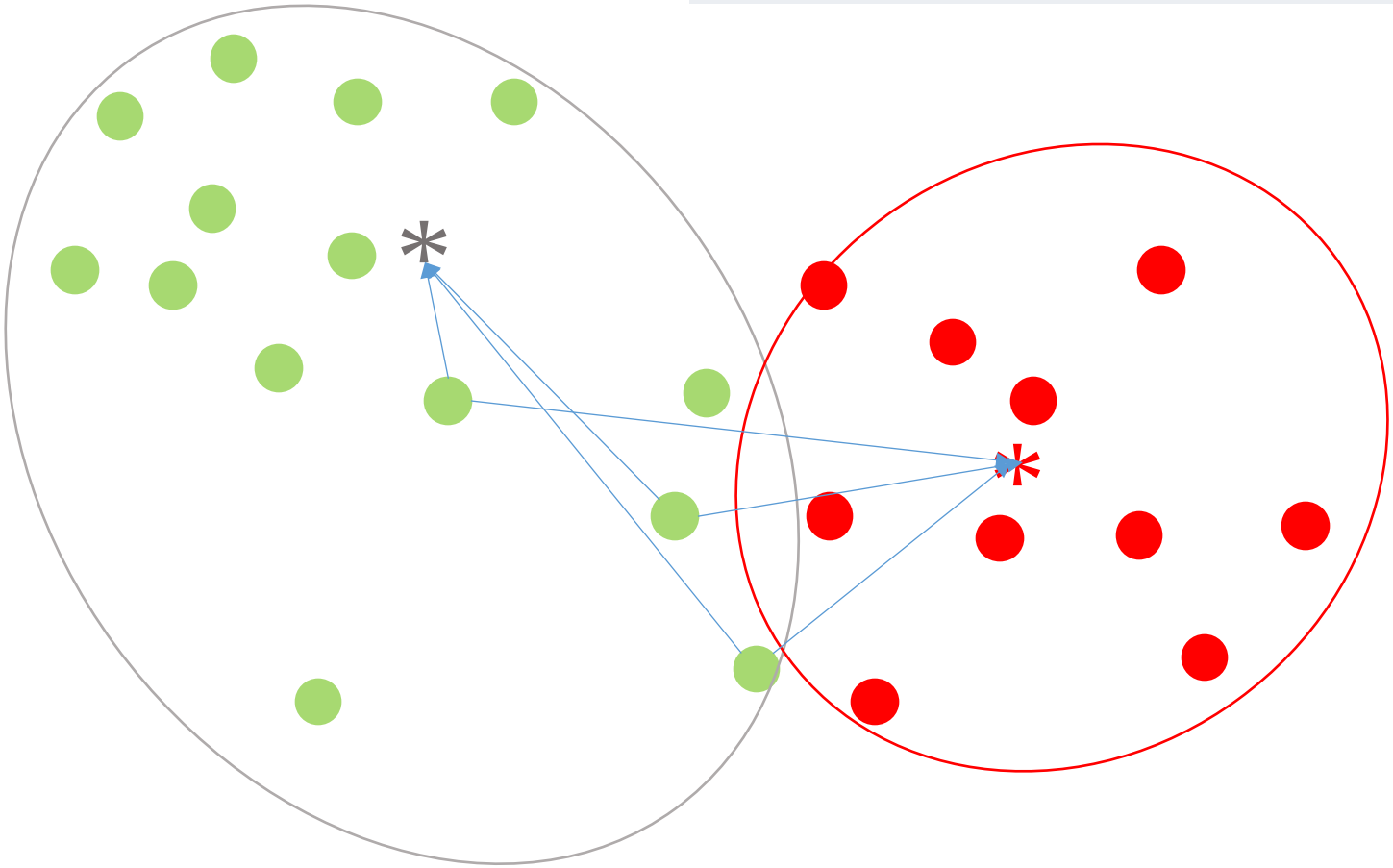
Clustering

- K-means Clustering

e.g. k =2

Then the following two steps are repeatedly carried out:

- Re-compute of the cluster midpoint.



Clustering

- K-means Clustering

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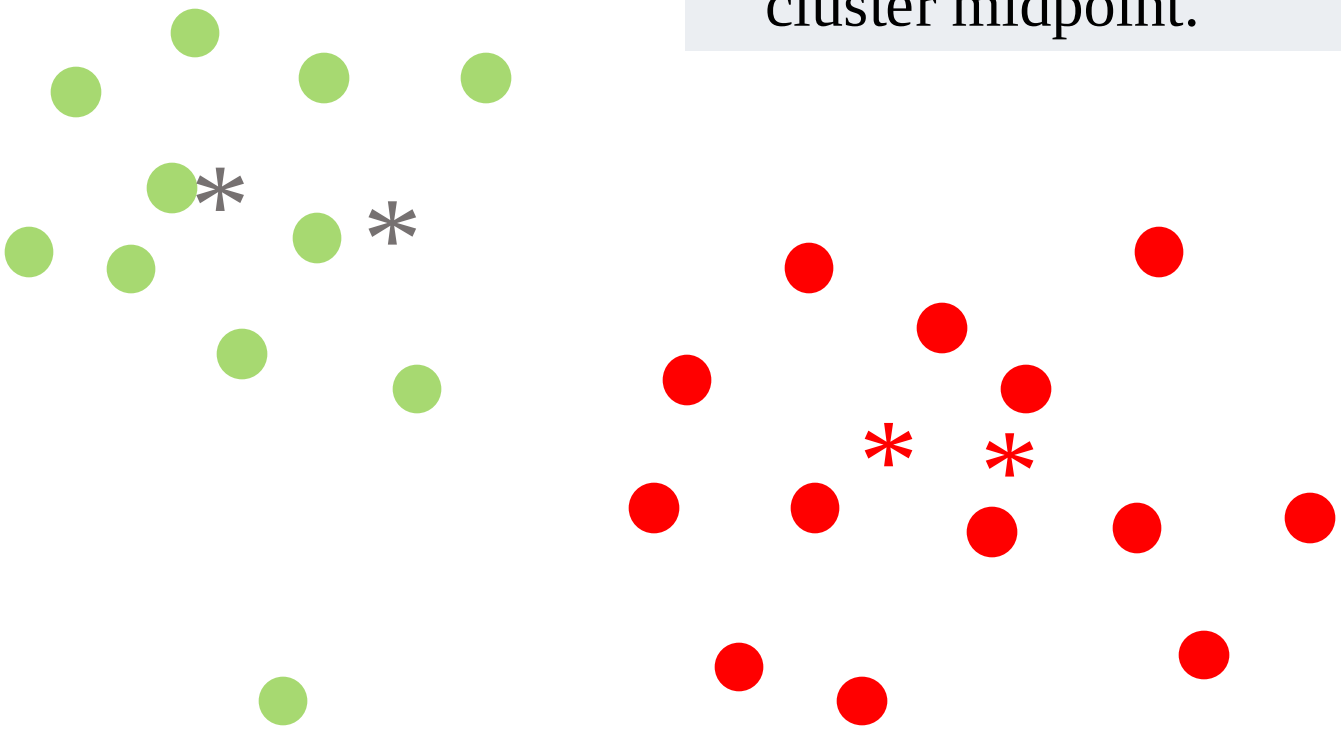

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Clustering

- K-means Clustering

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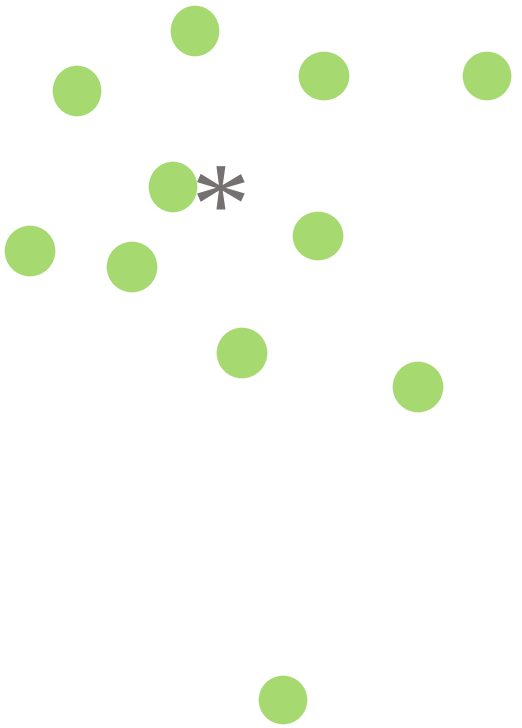

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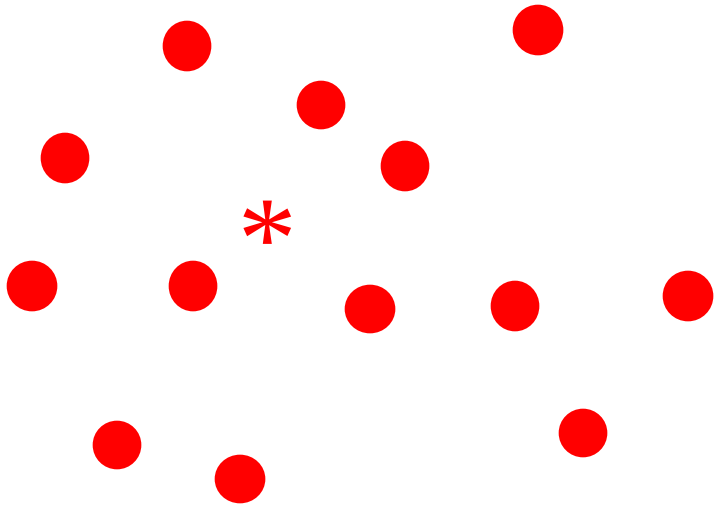


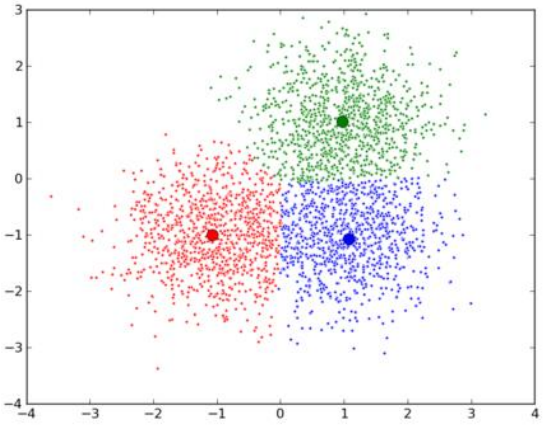
The algorithm converges

Clustering

K-means Clustering

The following two steps are repeatedly carried out:

- Initialize the midpoints
- Repeat the following 2 steps
  - Classify all data to their nearest cluster midpoint.
  - Re-compute of the cluster midpoint.
- Until the algorithm converges



Clustering

K-means Clustering -- Example

Try to cluster these samples X by k-means clustering, when k =2,

$$X = \begin{bmatrix} 0 & 0 & 1 & 5 & 5 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

- Initialize the midpoints
- Repeat the following 2 steps
  - Classify all data to their nearest cluster midpoint.
  - Re-compute of the cluster midpoint.
- Until the algorithm converges

Clustering

K-means Clustering -- Example

Try to cluster these samples X by k-means clustering, when k =2,

$$X = \begin{bmatrix} 0 & 0 & 1 & 5 & 5 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Step 1: First the *k* cluster midpoints $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k$ are randomly or manually initialized.

→ Suppose $m_1^{(0)} = (0,2)^T$ *is the mindpoint of cluster* $G_1^{(0)}$, $m_2^{(0)} = (0,0)^T$ *is the mindpoint of cluster* $G_2^{(0)}$

Clustering

K-means Clustering -- Example

Try to cluster these samples X by k-means clustering, when k =2,

$$X = \begin{bmatrix} 0 & 0 & 1 & 5 & 5 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Step 2: Classify all data to their nearest cluster midpoint.

→ Calculate the distance from $x_3 = (1,0)^T, x_4 = (5,\ 0)^T, x_5 = (5,2)^{\mathrm{T}}$ to the midpoints $m_1^{(0)}, m_2^{(0)}$:

$x_3 = (1,0)^T, d\left(x_3, m_1^{(0)}\right) = 5, d\left(x_3, m_2^{(0)}\right) = 1,$ So x_3 is $G_2^{(0)}$.

$x_4 = (5,5)^T, d\left(x_4, m_1^{(0)}\right) = 29, d\left(x_3, m_2^{(0)}\right) = 25,$ So x_4 is $G_2^{(0)}$.

$x_5 = (5,2)^{\mathrm{T}}, d\left(x_5, m_1^{(0)}\right) = 25, d\left(x_3, m_2^{(0)}\right) = 29,$ So x_5 is $G_1^{(0)}$.

Clustering

K-means Clustering -- Example

Try to cluster these samples X by k-means clustering, when k =2,

$$X = \begin{bmatrix} 0 & 0 & 1 & 5 & 5 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Step 3: New cluster $G_1^{(1)} = \{x_1, x_5\}$ and $G_2^{(1)} = \{x_2, x_3, x_4\}$. So, re-compute of the cluster midpoint.

$\rightarrow m_1^{(1)} = (2.5, 2.0)^T$ *is the mindpoint of cluster* $G_1^{(1)}$,

$m_2^{(1)} = (2, 0)^T$ *is the mindpoint of cluster* $G_2^{(1)}$

Clustering

K-means Clustering -- Example

Try to cluster these samples X by k-means clustering, when k =2,

$$X = \begin{bmatrix} 0 & 0 & 1 & 5 & 5 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Step 4: Repeat step 2 and step 3

→ Have new clusters $G_1^{(2)} = \{x_1, x_5\}$ and $G_2^{(2)} = \{x_2, x_3, x_4\}$.

Because the clusters is not change, the clustering stops! The final results is:

$$G_1^* = \{x_1, x_5\} \text{ and } G_2^* = \{x_2, x_3, x_4\}.$$

Clustering

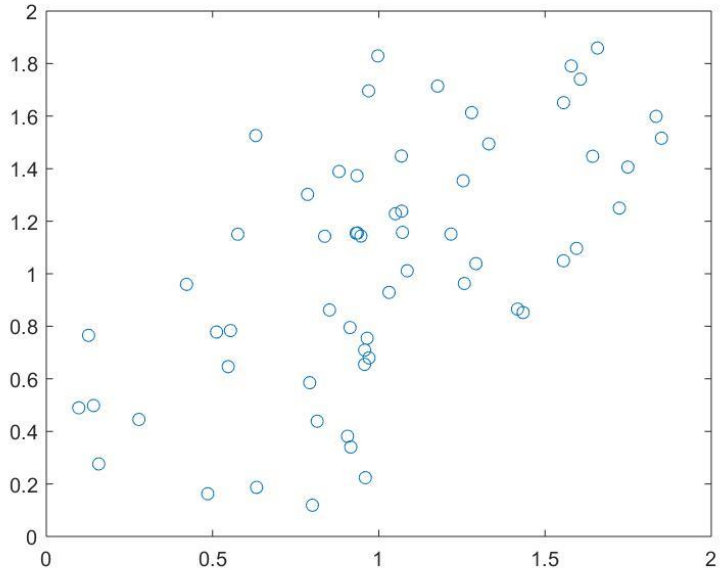
- K-means Clustering --- Example

X

2x60 double

	1	2	3	4	5	6	7	8	9	10
1	0.8147	0.9058	0.1270	0.9134	0.6324	0.0975	0.2785	0.5469	0.9575	0.9649
2	0.4387	0.3816	0.7655	0.7952	0.1869	0.4898	0.4456	0.6463	0.7094	0.7547

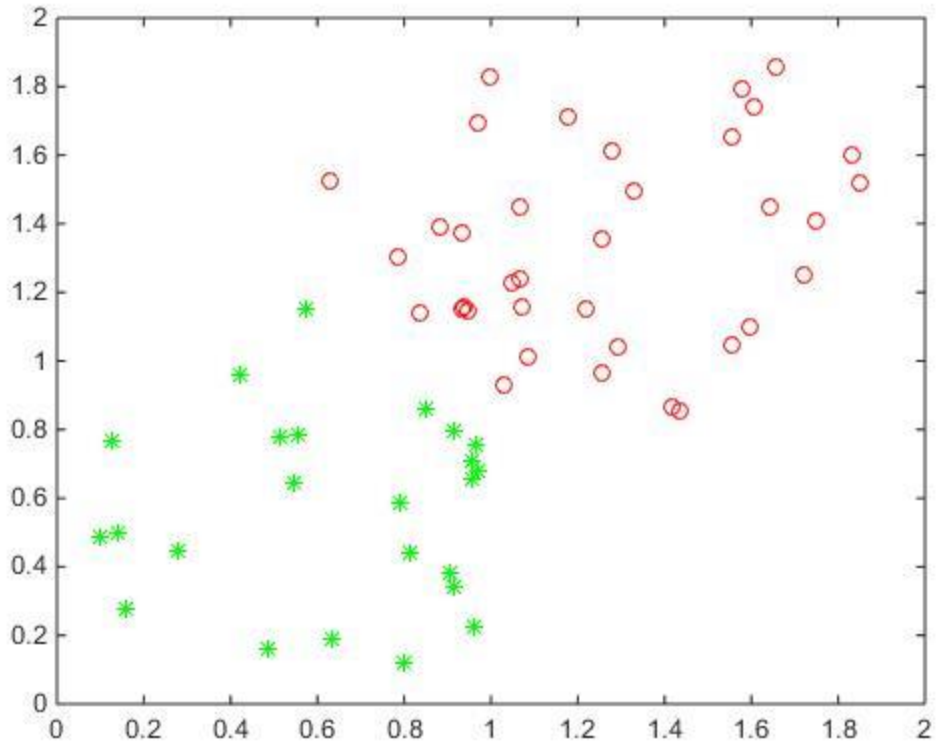
```
plot(X(1,:),X(2,:),'o')
```



Clustering

- K-means Clustering --- Example

```
IDX = kmeans(X,2);
…… % plot the results
```



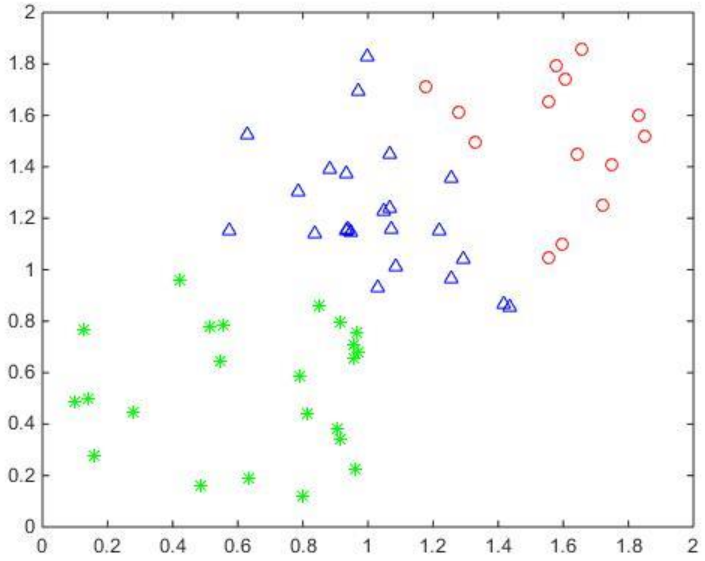
Clustering

K-means Clustering --- Example

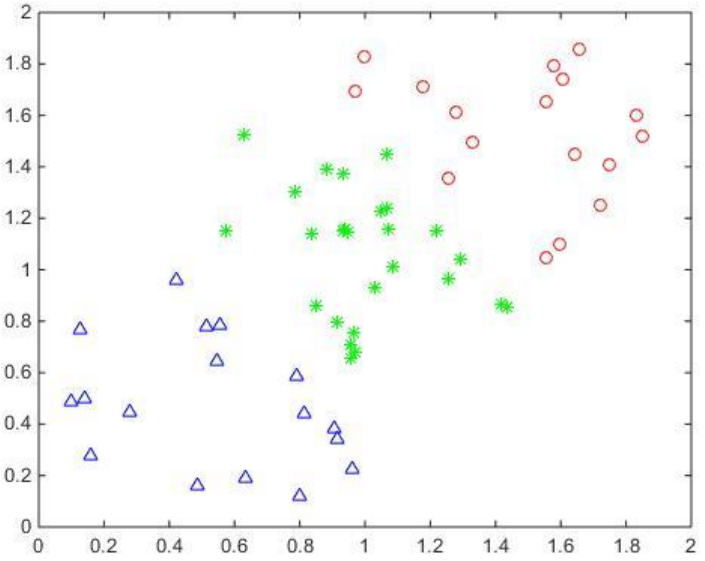
```
IDX = kmeans(X,3);
…… % plot the results
```

The results are different!
Why?

Result 1

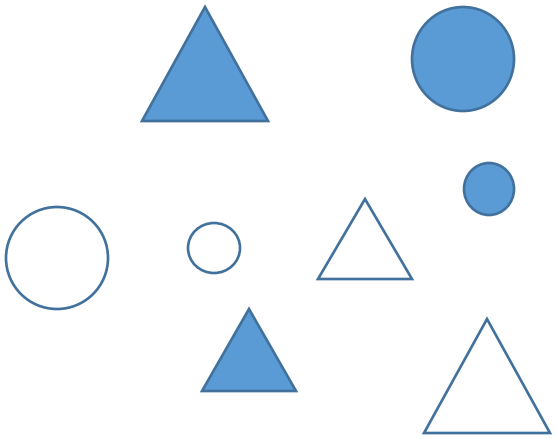


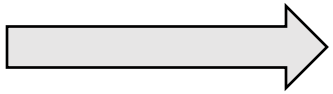
Result 2



Clustering

- K-means Clustering -Example





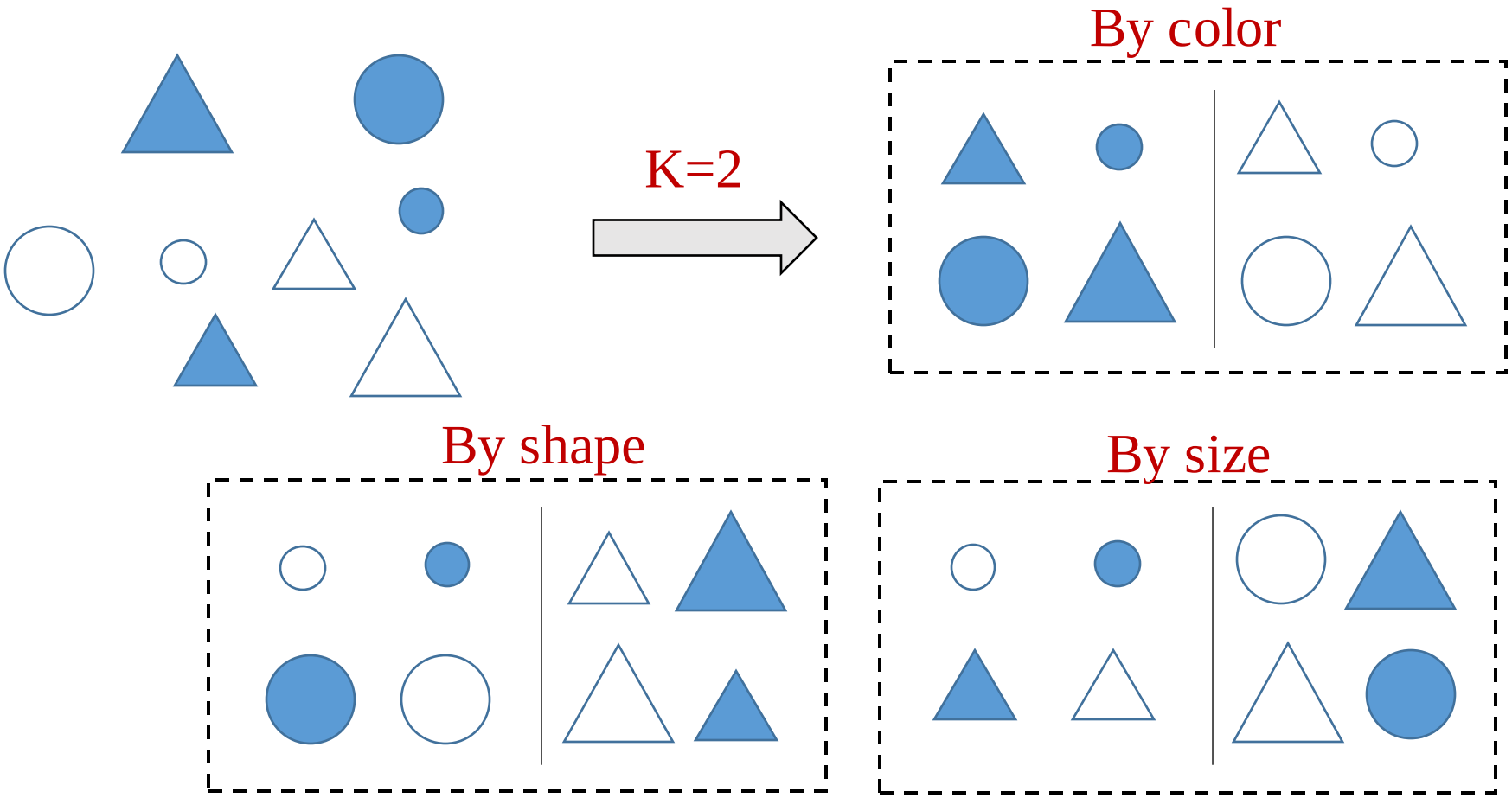
Clustering

K=2?

K=4?

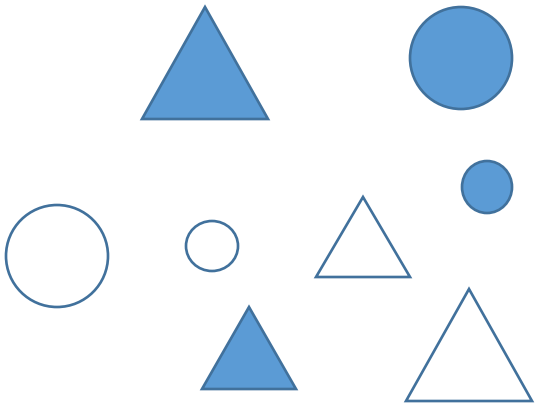
Clustering

- K-means Clustering -Example

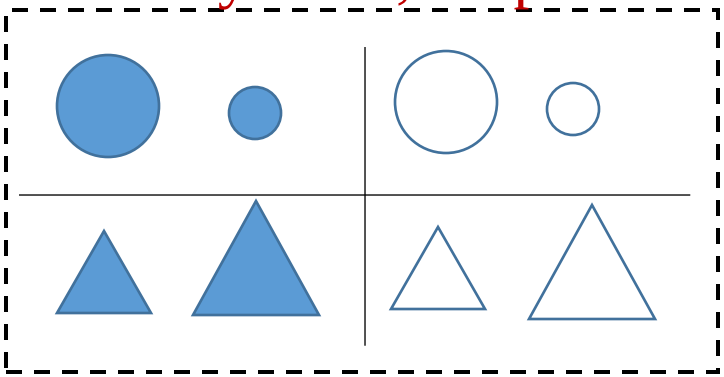


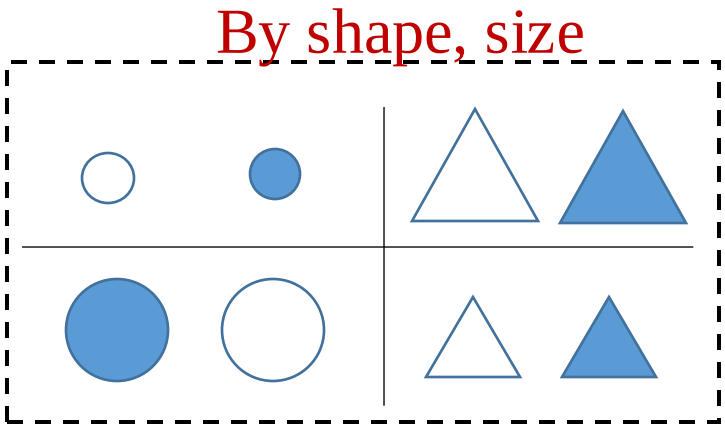
Clustering

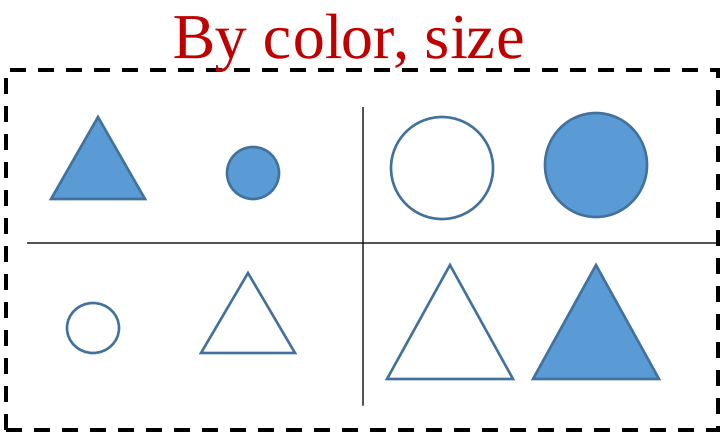
- K-means Clustering -Example



K=4







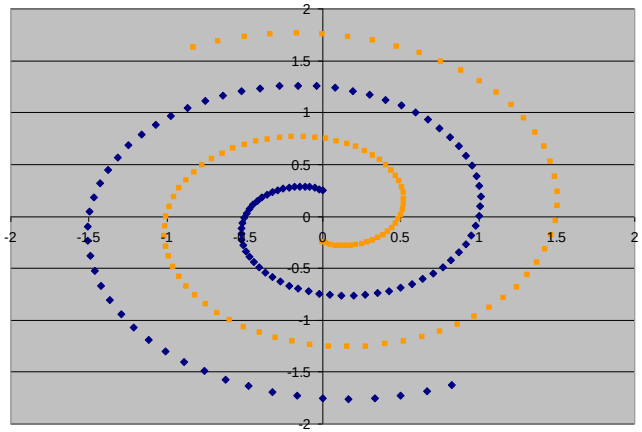
Clustering

K-means Clustering --- Disadvantage

- 1. Time complexity:
  - O(NKT), where N is the number of data, K is the number of clusters, and T is the number of iterations.
- 2. Sensitive to noise
- 3. Different results are obtained with different initial centers.

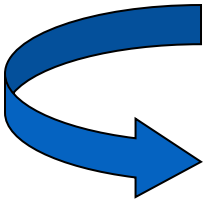
Clustering

K-means Clustering --- Example

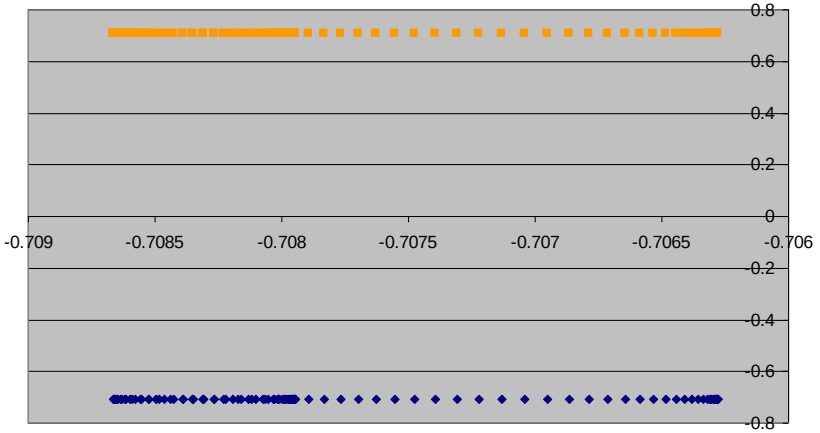


Dataset exhibits complex cluster shapes

$\Rightarrow$ K-means performs very poorly in this space due bias toward dense spherical clusters.



In the embedded space given by two leading eigenvectors, clusters are trivial to separate.



Clustering

- Spectral Clustering
  - Algorithms that cluster points using eigenvectors of matrices derived from the data.
  - Obtain data representation in the low-dimensional space that can be easily clustered.
  - Variety of methods that use the eigenvectors differently
  - Disadvantage: difficult to understand….

Clustering

- Spectral Clustering

- Three basic stages:
  1. Pre-processing
     - Construct a matrix representation of the dataset.
  2. Decomposition
     - Compute eigenvalues and eigenvectors of the matrix.
     - Map each point to a lower-dimensional representation based on one or more eigenvectors.
  3. Grouping
     - Assign points to two or more clusters (e.g. by k means method), based on the new representation.

Clustering

- Spectral Clustering

1 . Pre-processing

- Construct a matrix representation of the dataset. (Build Laplacian matrix ***L)***

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1.5	-0.8	-0.6	0	-0.1	0
x_2	-0.8	1.6	-0.8	0	0	0
x_3	-0.6	-0.8	1.6	-0.2	0	0
x_4	0	0	-0.2	1.7	-0.8	-0.7
x_5	-0.1	0	0	-0.8	1.7	-0.8
x_6	0	0	0	-0.7	-0.8	1.5

2. Decomposition

– Find eigenvalues Λ and eigenvectors X of the matrix ***L***

– Map vertices to corresponding components of smallest two eigenvalues.

$\Lambda =$

0.0
0.4
2.2
2.3
2.5
3.0

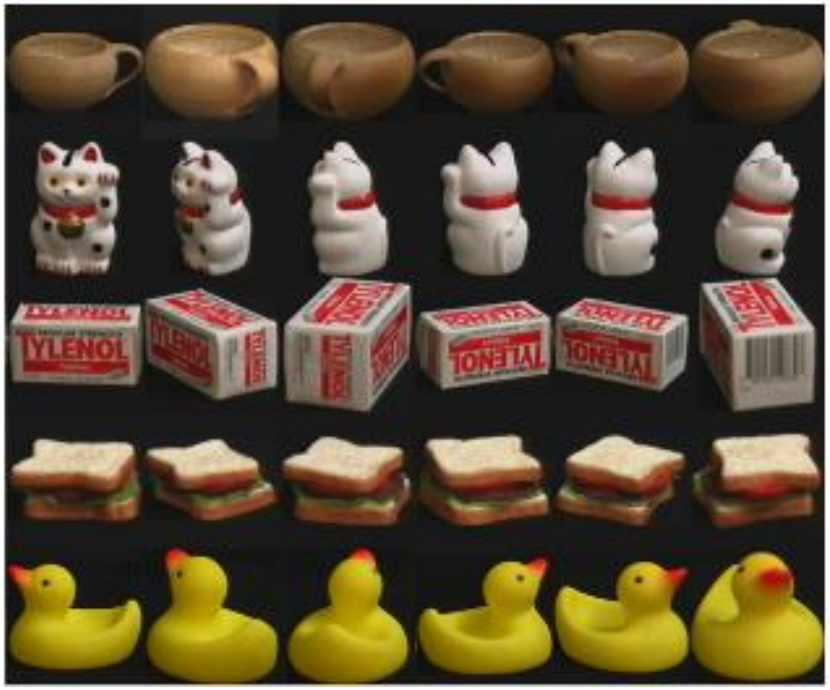
$X =$

0.4	0.2	0.1	0.4	-0.2	-0.9
0.4	0.2	0.1	-0.	0.4	0.3
0.4	0.2	-0.2	0.0	-0.2	0.6
0.4	-0.4	0.9	0.2	-0.4	-0.6
0.4	-0.7	-0.4	-0.8	-0.6	-0.2
0.4	-0.7	-0.2	0.5	0.8	0.9

It is easier to divide these six points into two clusters using this new representation.

Clustering

- Spectral Clustering—Applications: image clustering



Clustering

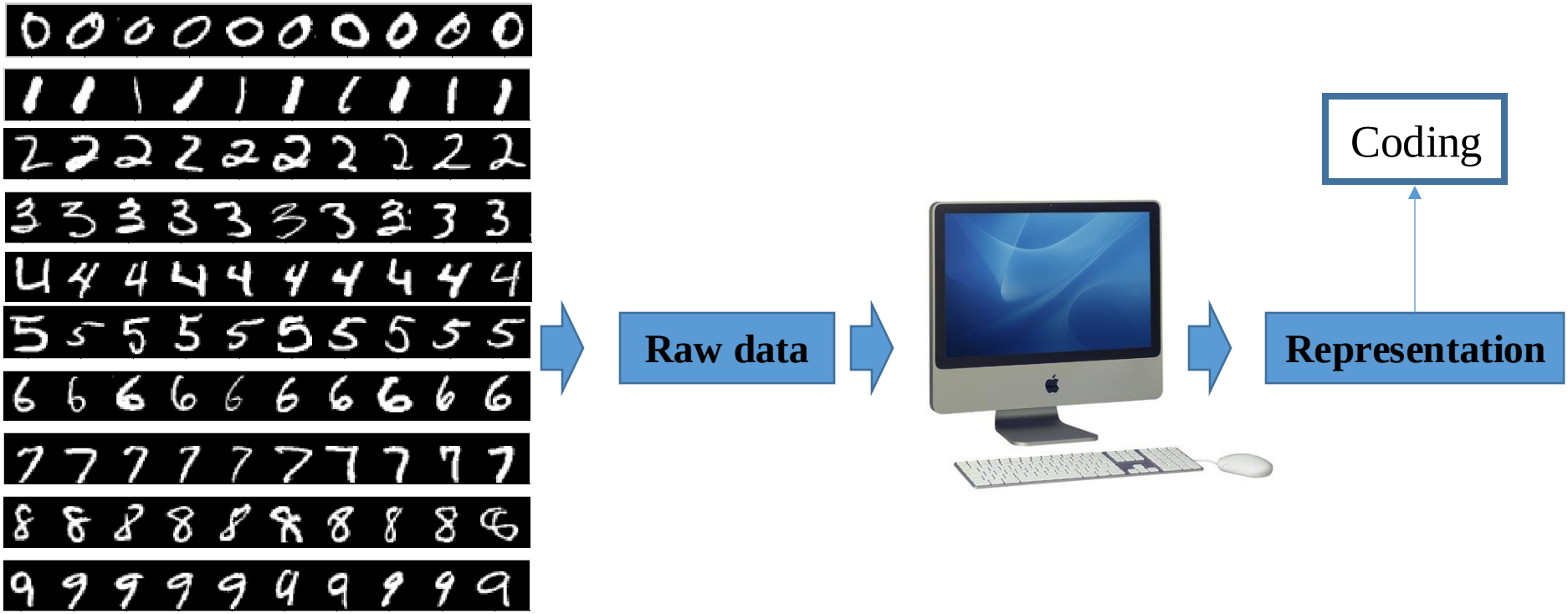
- Spectral Clustering ---Applications: Motion segmentation

Unsupervised learning

- Clustering
  - K-means method
  - Spectral clustering
- Representation learning

Representation Learning

- Representation = re + presentation



Unsupervised learning

- Clustering
  - K-means method
  - Spectral clustering
- Representation learning
  - *Linear coding*
  - PCA
  - Applications

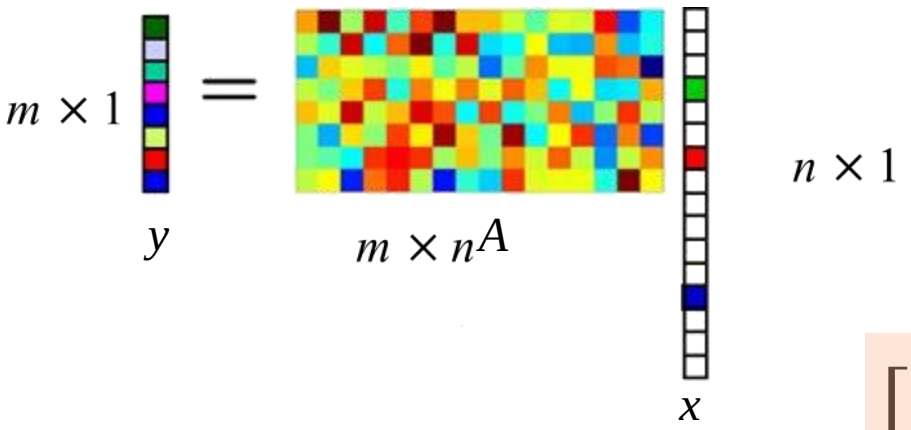
Representation Learning

- Linear coding

representation

A: “dictionary”

$$y = Ax \quad \in \mathbb{R}^m$$



e.g.

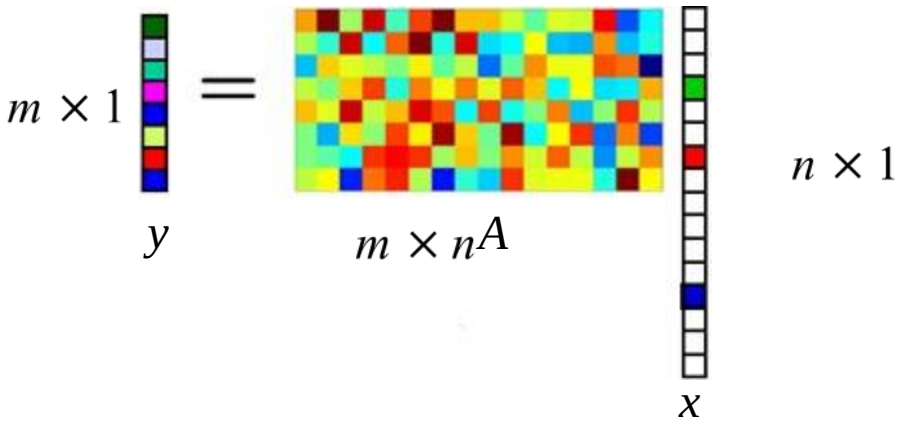
$$y = \begin{bmatrix} 0.75 \\ 0.27 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0.1 & 0 & 0 \\ 0.2 & 0.1 & 0 & 0.1 & 1 & 0 \\ 0 & 0.1 & 1 & 0.2 & 0.1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.6 \\ 0 \\ 0 \\ 1.5 \\ 0 \\ 0 \end{bmatrix}$$

Representation Learning

- Linear coding

representation

$$y = Ax \in R^m$$



Considering the solution to the linear equation:

If m>n => overdetermined => no solution / unique solution

If m<n => underdetermined => no solution / infinite solutions

Unsupervised learning

- Clustering
  - K-means method
  - Spectral clustering
- Representation learning
  - Linear coding
  - *PCA*
  - Applications

Representation Learning

- Linear Feature Extraction

➢ Given the original d-dimension feature space $X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m) \in \mathbb{R}^{d \times m}$

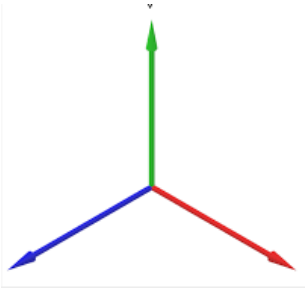
➢ Get the reduced d'-dimension feature space $Z = (\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_m) \in \mathbb{R}^{d' \times m}$ after transformation ($d' < d$)

➢ Transformation process:

$$Z = \boldsymbol{W}^T X$$

Where $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}) \in \mathbb{R}^{d \times d'}$ is the transformation matrix, $\boldsymbol{w}_i \in \mathbb{R}^{d \times 1}$, and $Z \in \mathbb{R}^{d' \times m}$ is the coordinate expression of sample X in low dimension space.

➢ If $\boldsymbol{w}_i^T \boldsymbol{w}_j = 0 (\mathrm{i} \neq j)$, then $\boldsymbol{w}_i$ is orthogonal to $\boldsymbol{w}_j$ ($\boldsymbol{w}_i$ is independent from $\boldsymbol{w}_j$), the new coordinate system $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}\}$ is orthogonal, and $\boldsymbol{W}$ is the orthogonal matrix.



Representation Learning

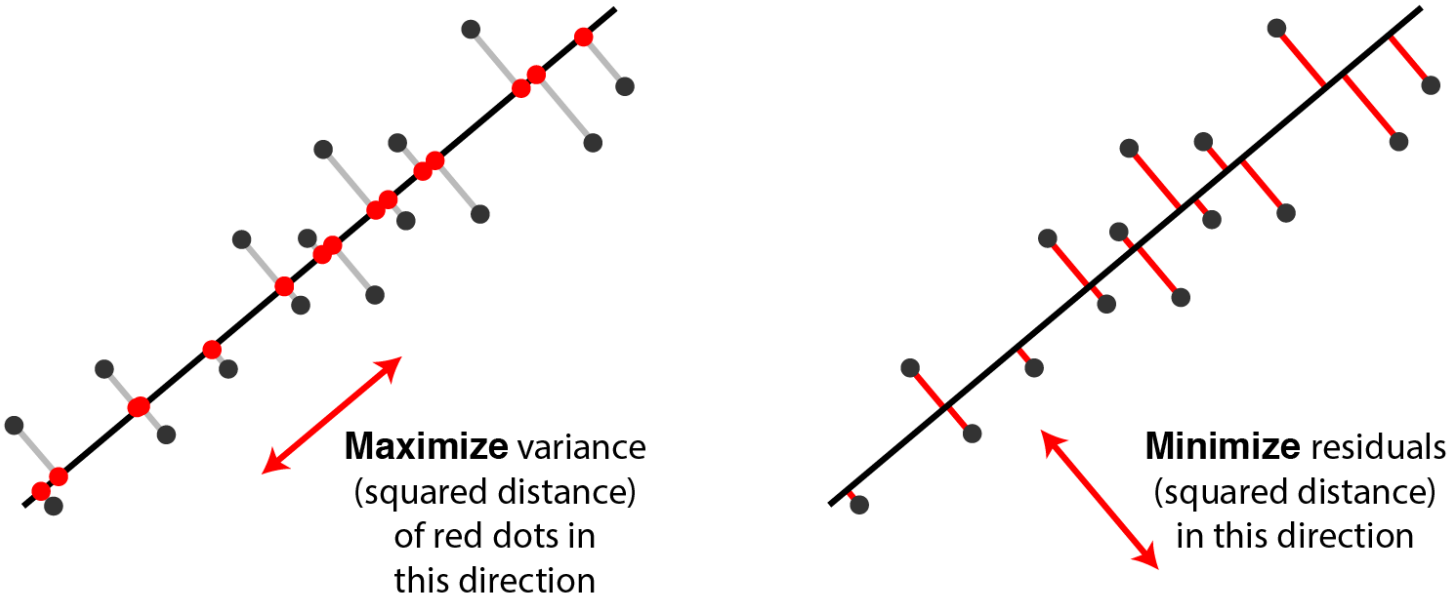
Principal Component Analysis (PCA)

- Linear combination of original features: use the orthogonal transform matrix $\boldsymbol{W}$ to transform the d relevant features into d' $(d' < d)$ irrelevant features. These d' irrelevant features are called principal components for classification.
- Use principal components to approximate the original sample.
- Realize the dimension reduction by replace original sample using few principal components.

Representation Learning

Principal Component Analysis (PCA)

- **Minimize reconstruction error (residual):** the sample $\tilde{\boldsymbol{x}}$ reconstructed from the reduced (projected) space is close enough to the original sample $\boldsymbol{x}$.
- **Maximum class separability (variance):** ensure that projected data from different classes can be separated well.



Representation Learning

Principal Component Analysis (PCA)

- Minimize reconstruction error

➢ Data standardization. Centralization the original data (subtract mean vector). That is, set $\boldsymbol{x}_i = \boldsymbol{x}_i - \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{x}_i$, then $\sum_{i=1}^{m} \boldsymbol{x}_i = 0$, $\boldsymbol{x}_i \in \mathbb{R}^{d\times 1}$.

➢ Assume transformation matrix $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}) \in \mathbb{R}^{d\times d'}$, $\boldsymbol{w}_i \in \mathbb{R}^{d\times 1}$ is the standard orthogonal basis vector. That is, $\|\boldsymbol{w}_i\|_2 = 1, \boldsymbol{W}^T\boldsymbol{W} = \boldsymbol{I}$, $\boldsymbol{w}_i^T\boldsymbol{w}_j = 0 (i \neq j)$.

➢ The projection of $\boldsymbol{x}_i$ in low dimension coordinate system is $\boldsymbol{z}_i = \boldsymbol{W}^T\boldsymbol{x}_i$, $\boldsymbol{z}_i = (z_{i1}; z_{i2}; \dots; z_{id'}) \in \mathbb{R}^{d'\times 1}$. Where $z_{ij} = \boldsymbol{w}_j^T\boldsymbol{x}_i$ is the jth-coordinate of $\boldsymbol{x}_i$ in low dimension space.

➢ Reconstruct $\boldsymbol{x}_i$ by $\boldsymbol{z}_i$, then $\tilde{\boldsymbol{x}}_i = \sum_{j=1}^{d'} z_{ij}\boldsymbol{w}_j$

Minimize reconstruction error : $\tilde{\boldsymbol{x}}_i$, $\boldsymbol{x}_i$

Representation Learning

Principal Component Analysis (PCA)

- Minimize reconstruction error

➢For the entire training set, the distance between original sample $\boldsymbol{x}_i$ and reconstructed sample $\tilde{\boldsymbol{x}}_i$ is

$$\sum_{i=1}^{m} \|\tilde{\boldsymbol{x}}_i - \boldsymbol{x}_i\|_2^2 = -\sum_{i=1}^{m} \boldsymbol{z}_i^T \boldsymbol{z}_i + \sum_{i=1}^{m} \boldsymbol{x}_i^T \boldsymbol{x}_i$$

$$\propto -tr\left(\boldsymbol{W}^T \left(\sum_{i=1}^{m} \boldsymbol{x}_i \boldsymbol{x}_i^T\right) \boldsymbol{W}\right)$$

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

➢Take minimizing the reconstruction distance as the objective function, since $\sum_{i=1}^{m} \boldsymbol{x}_i \boldsymbol{x}_i^T$ is the covariance matrix $\boldsymbol{X}\boldsymbol{X}^T$, then the objective function is

$$\min_{\boldsymbol{W}} -tr(\boldsymbol{W}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W})$$

$$s.t. \boldsymbol{W}^T \boldsymbol{W} = \boldsymbol{I}$$

Representation Learning

Principal Component Analysis (PCA)

- Proof

$$\sum_{i=1}^{m}\|\hat{\boldsymbol{x}}_i-\boldsymbol{x}_i\|_2^2=\sum_{i=1}^{m}\left\|\sum_{j=1}^{d'}z_{ij}\boldsymbol{w}_j-\boldsymbol{x}_i\right\|_2^2=\sum_{i=1}^{m}\|\boldsymbol{W}\boldsymbol{z}_i-\boldsymbol{x}_i\|_2^2$$

$$=\sum_{i=1}^{m}(\boldsymbol{W}\boldsymbol{z}_i-\boldsymbol{x}_i)^T(\boldsymbol{W}\boldsymbol{z}_i-\boldsymbol{x}_i)$$

$$=-\sum_{i=1}^{m}\boldsymbol{z}_i^T\boldsymbol{z}_i+\sum_{i=1}^{m}\boldsymbol{x}_i^T\boldsymbol{x}_i$$

$$=-\sum_{i=1}^{m}tr\left(\boldsymbol{z}_i\boldsymbol{z}_i^T\right)+\text{const}$$

$$=-tr\left(\sum_{i=1}^{m}\boldsymbol{z}_i\boldsymbol{z}_i^T\right)+\text{const}$$

$$=-tr\left(\sum_{i=1}^{m}\boldsymbol{W}^T\boldsymbol{x}_i\boldsymbol{x}_i^T\boldsymbol{W}\right)+\text{const}$$

$$\approx-tr\left(\boldsymbol{W}^T\left(\sum_{i=1}^{m}\boldsymbol{x}_i\boldsymbol{x}_i^T\right)\boldsymbol{W}\right)+\text{const}$$

$$\propto-tr\left(\boldsymbol{W}^T\left(\sum_{i=1}^{m}\boldsymbol{x}_i\boldsymbol{x}_i^T\right)\boldsymbol{W}\right)$$

Representation Learning

- Principal Component Analysis (PCA)
  - Maximum class separability

$$\max_{\boldsymbol{W}} tr(\boldsymbol{W}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W})$$
$$s.t. \boldsymbol{W}^T \boldsymbol{W} = \boldsymbol{I}$$

$$\min_{\boldsymbol{W}} -tr(\boldsymbol{W}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W})$$
$$s.t. \boldsymbol{W}^T \boldsymbol{W} = \boldsymbol{I}$$

Representation Learning

Principal Component Analysis (PCA)

Solve PCA to get $\boldsymbol{W = (w_1, w_2, \dots, w_{d'})}$

$$\min_{\boldsymbol{W}} -tr(\boldsymbol{W}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W})$$
$$s.t. \boldsymbol{W}^T \boldsymbol{W} = \boldsymbol{I}$$

Where $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m) \in \mathbb{R}^{d \times \boldsymbol{m}}, \boldsymbol{W} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}\} \in \mathbb{R}^{d \times d'}, \boldsymbol{I} \in \mathbb{R}^{d' \times d'}$

- **S1.** Define the Lagrange function using Lagrange multiplier matrix $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \dots, \lambda_{d'}) \in \mathbb{R}^{d' \times d'}$, $\boldsymbol{\Lambda}$ is the diagonal matrix.

$$L(\boldsymbol{W}, \boldsymbol{\Lambda}) = -tr(\boldsymbol{W}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W}) + tr\left(\boldsymbol{\Lambda}^T (\boldsymbol{W}^T \boldsymbol{W} - \boldsymbol{I})\right)$$

- **S2.** Set the partial of $L(\boldsymbol{W}, \boldsymbol{\Lambda})$ on $\boldsymbol{W}$ as 0.

$$\frac{\partial L(\boldsymbol{W}, \boldsymbol{\Lambda})}{\partial \boldsymbol{W}} = -2\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W} + 2\boldsymbol{W}\boldsymbol{\Lambda} = \boldsymbol{0}$$

$$\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{W} = \boldsymbol{W}\boldsymbol{\Lambda}$$

$$\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{w}_i = \lambda_i \boldsymbol{w}_i$$

The definition of eigenvalue and eigenvector for matrix. Then $\boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{W}\boldsymbol{\Lambda}\boldsymbol{W}^T$

Solve eigenvalue $\boldsymbol{\Lambda}$ and corresponding eigenvector $\boldsymbol{W}$ for matrix $\boldsymbol{X}\boldsymbol{X}^T$

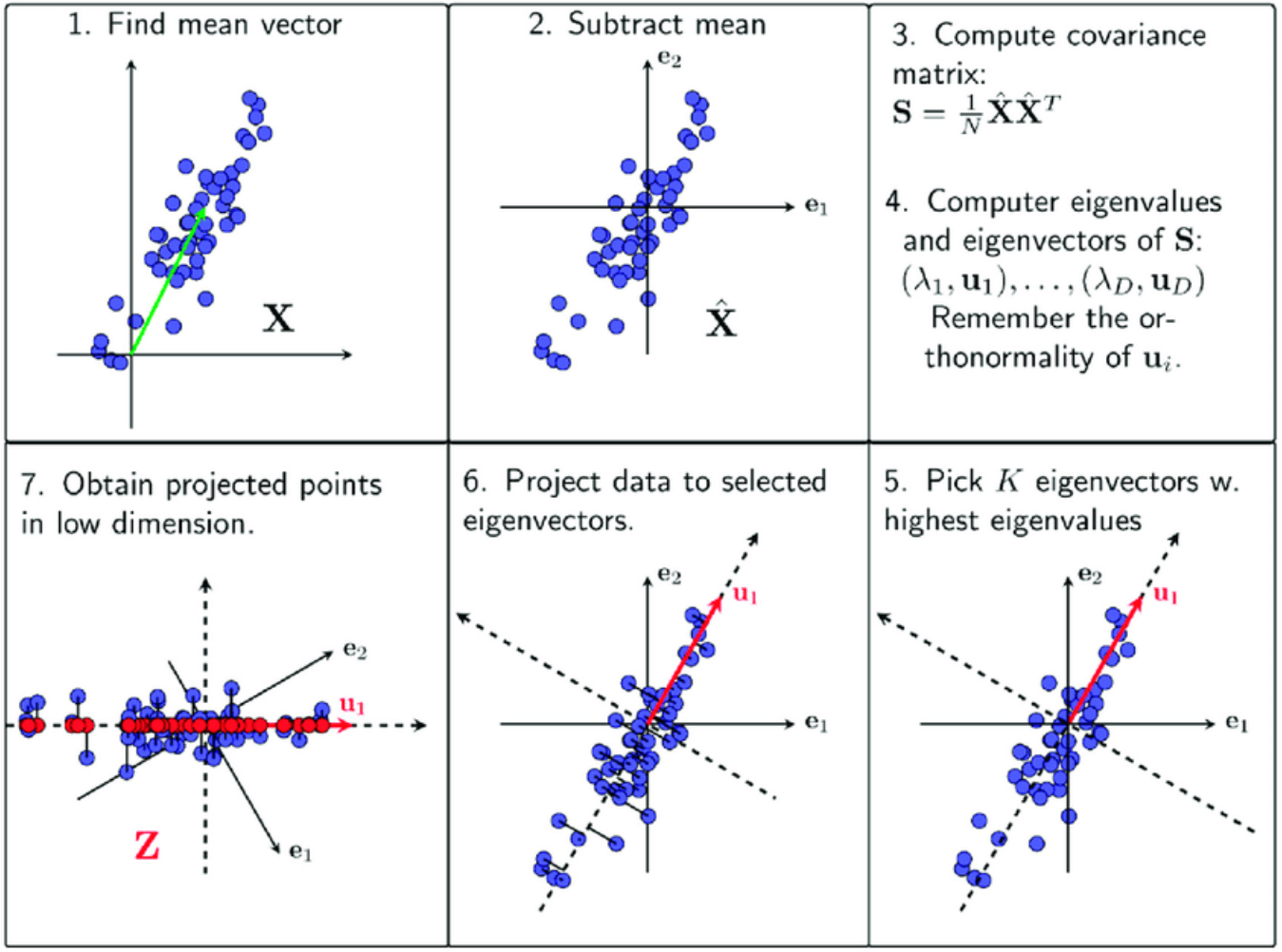
Representation Learning

Principal Component Analysis (PCA)

- Process to implement dimension reduction using PCA
- **S1:** Standardization. $\boldsymbol{x}_i = \boldsymbol{x}_i - \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{x}_i$
- **S2:** Compute the covariance matrix $\frac{\mathbf{1}}{m}\boldsymbol{X}\boldsymbol{X}^T$
- **S3:** Eigenvalue decompose the matrix $\frac{\mathbf{1}}{m}\boldsymbol{X}\boldsymbol{X}^T$
- **S4:** Take the top maximum d' eigenvalues and corresponding eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{d'}$
- **S5:** Get the transformation matrix $\boldsymbol{W} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{d'}\}$, and output the reduced feature vectors $\boldsymbol{z}_i$ for original samples.

Representation Learning

- Principal Component Analysis (PCA)



Representation Learning

Principal Component Analysis (PCA)

Example

- Problem: Given the original sample set $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5) = \begin{pmatrix} -1 & -1 & 0 & 2 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix}$. Reduce sample from 2-D to 1-D using PCA.

- Process to implement dimension reduction using PCA
- **S1:** Standardization. $\boldsymbol{x}_i = \boldsymbol{x}_i - \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{x}_i$
- **S2:** Compute the covariance matrix $\frac{\mathbf{1}}{m}\boldsymbol{X}\boldsymbol{X}^T$
- **S3:** Eigenvalue decompose the matrix $\frac{\mathbf{1}}{m}\boldsymbol{X}\boldsymbol{X}^T$
- **S4:** Take the top maximum d' eigenvalues and corresponding eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}$
- **S5:** Get the transformation matrix $\boldsymbol{W} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{d'}\}$, and output the reduced feature vectors $\boldsymbol{z}_i$ for original samples.

Representation Learning

Answer:

- S1. Standardization. The mean vector $\frac{1}{m}\sum_{i=1}^{m}\boldsymbol{x}_i = 0$
- S2. Compute covariance matrix $\boldsymbol{A} = \frac{1}{5}\boldsymbol{X}\boldsymbol{X}^T = \frac{1}{5}\begin{pmatrix}6 & 4\\ 4 & 6\end{pmatrix}$
- S3. Eigenvalue decompose.

(1) Solve Eigenvalue by $|\lambda I - \boldsymbol{A}| = 0$

$$|\lambda I - \boldsymbol{A}| = \begin{vmatrix}\lambda - \frac{6}{5} & -\frac{4}{5}\\ -\frac{4}{5} & \lambda - \frac{6}{5}\end{vmatrix} = (\lambda - \frac{6}{5})^2 - \frac{16}{25} = (\lambda - 2)\left(\lambda - \frac{2}{5}\right) = 0$$

$$\lambda_1 = 2,\ \ \lambda_2 = \frac{2}{5}$$

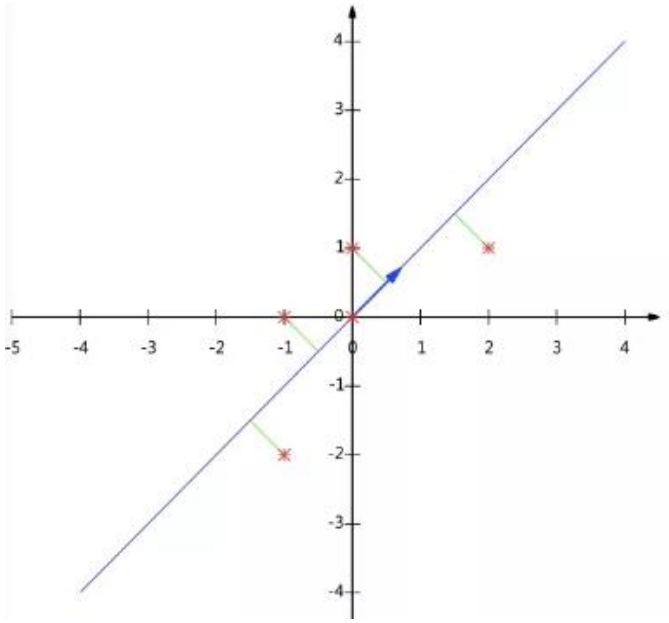
(2) Solve Eigenvector by $(\lambda I - A)\boldsymbol{w} = \boldsymbol{0}$

$\lambda_1 = 2 \rightarrow \boldsymbol{w}_1 = \binom{1}{1}, \quad \lambda_2 = \frac{2}{5} \rightarrow \boldsymbol{w}_2 = \binom{-1}{1}$

Representation Learning

- S4. Rank the eigenvalues, and select top $d' = 1$ eigenvalue λ_1 and its corresponding eigenvector $\boldsymbol{w}_1$. Standardization eigenvector to get $\boldsymbol{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
- S5. Dimension reduction.

$$Z = W^T X = \left(-\frac{3}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \quad 0 \quad \frac{3}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)$$



Unsupervised learning

- Clustering
  - K-means method
  - Spectral clustering
- Representation learning
  - Linear coding
  - PCA
  - Others
  - Applications

Representation Learning

- Applications

Facial Image Compression

Source image

JPEG image

JPEG2000 image

K-SVD image

Representation Learning

- Applications

Image Deblurring

Source image

Blurred image

After deblurring

Representation Learning

- Applications

Image Denoising

Source image

Noisy image

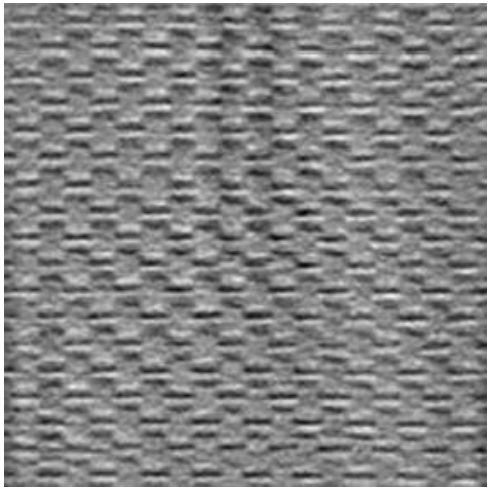
Denoising result

Representation Learning

- Applications

Morphological Component Analysis

Source image



Texture component

Cartoon component

Representation Learning

- Applications

Image Inpainting

Source image

Degraded image

Inpainting result

Representation Learning

Have a try…

- *Generate some points in two-dimensional plane.*
- *Try to design the k-means clustering algorithm or spectral clustering algorithm to cluster these points.*
- *Compare the results.*

Machine Learning

Supervised learning

Unsupervised learning

Reinforcement learning

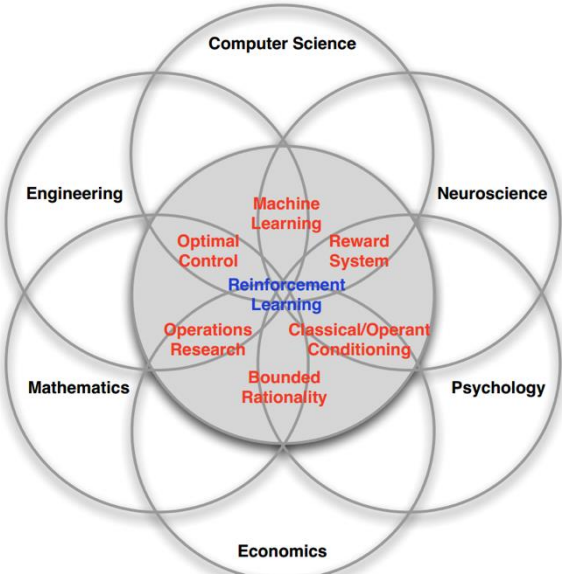
Introduction to Reinforcement learning

- Reinforcement learning
- Q-Learning

Reinforcement learning

Reinforcement Learning

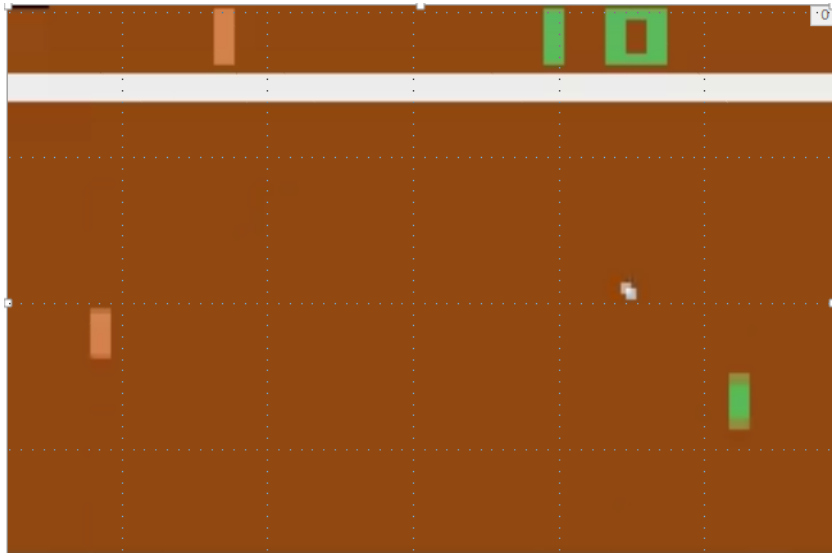
- "AI=RL" by David Silver
- Agent-oriented learning—learning by interacting with an environment to achieve a goal
- Learning by trial and error, with only delayed evaluative feedback (reward)



1. Different ML methods

- Reinforcement Learning

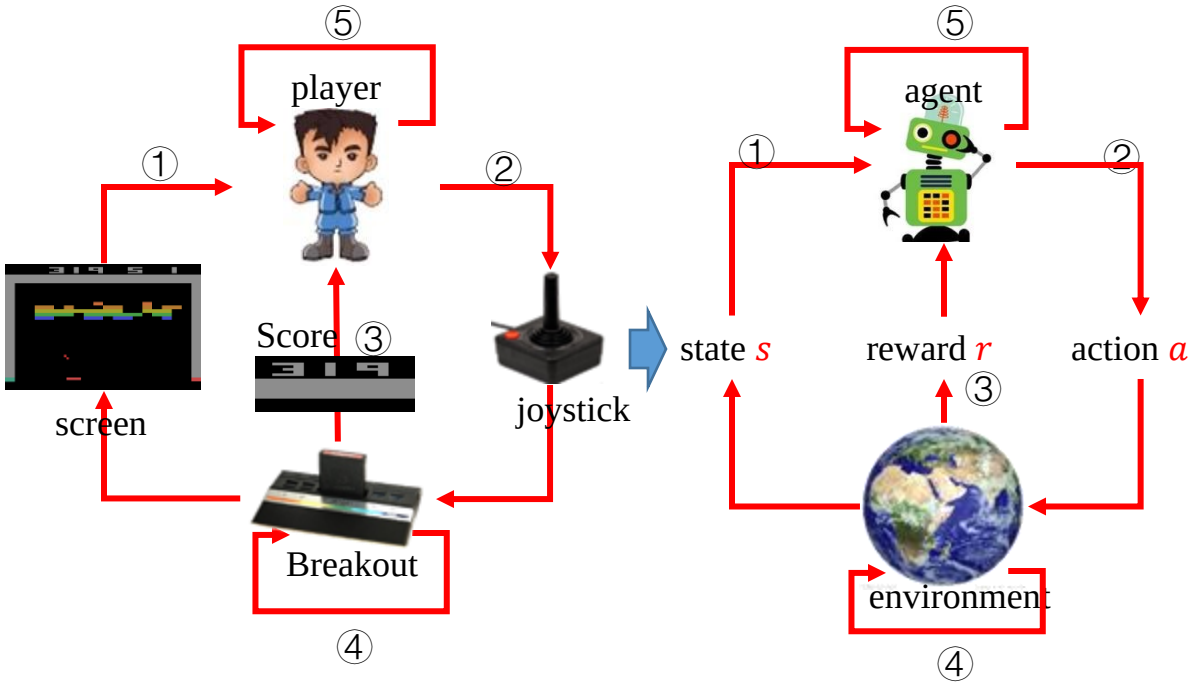
  - **Game Pong**



  - **Game Breakout**

1. Different ML methods

Reinforcement Learning



- Rules are unknown
- Learn directly from the interaction

At each time step t:

① Agent receives state $\boldsymbol{s(t)}$

② Agent executes an action $\boldsymbol{a(t)}$ by his action policy $\boldsymbol{\pi(s(t))}$

③ Environment emits a immediate reward $\boldsymbol{r(t+1)}$ to agent

④ Environment changes its state to $\boldsymbol{s(t+1)}$

⑤ Agent improves his policy $\boldsymbol{\pi(s)}$ according to the reward.

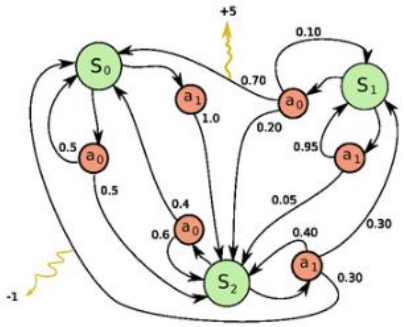
$$\begin{cases} < s, a, r, s' > \\ s \leftarrow s' \end{cases}$$

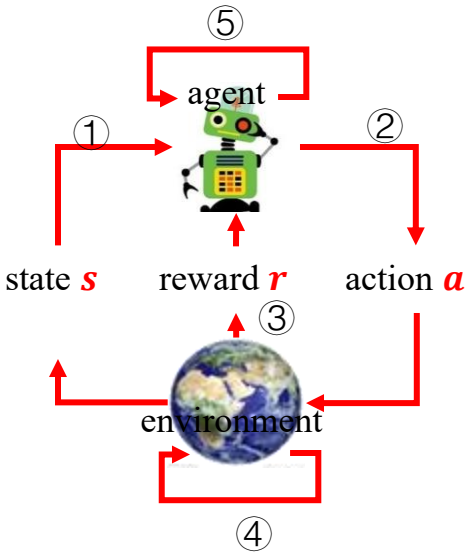
Reinforcement learning

- RL problem can be described as a Markov decision process
  - The future is independent of the past given the present
- One episode of this process forms a finite sequence :

$$s(0), a(0), r(1), s(1), a(1), r(2), \cdots\cdots, s(n-1), a(n-1), r(n), s(n)$$

$$\begin{cases} < s, a, r, s' > \\ s \leftarrow s' \end{cases}$$





- The agent are always trying to get the maximum rewards through policy $\pi(s)$

Question: How to define the maximum reward ?

Reinforcement learning

One episode of this process forms a finite sequence of states, actions, and rewards:

$$s(0), a(0), r(1), s(1), a(1), r(2), \cdots\cdots, s(n-1), a(n-1), r(n), s(n)$$

- Total reward of one episode:

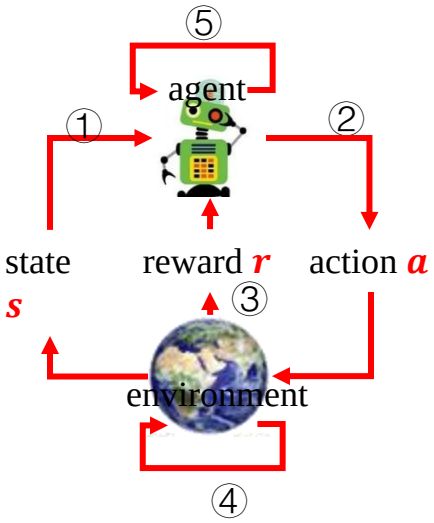
$$R = r(1) + r(2) + r(3) + \cdots\cdots r(n-1) + r(n)$$

- Total future reward from time step t :

$$R(t)$$
$$= r(t) + r(t+1) + r(t+2) + \cdots\cdots r(n-1) + r(n)$$

- Discounted future reward reward from time step t :

$$R(t) = r(t) + \gamma r(t+1) + \gamma^2 r(t+2) + \cdots\cdots + \gamma^{n-t} r_{\mathrm{n}}$$



Question: How can agent get the maximum reward ?

Reinforcement learning

Question: How can agent get the maximum reward ?

$$R = r(1) + r(2) + r(3) + \cdots\cdots r(n-1) + r(n)$$
$$= r(1) + r(2) + r(3) + \cdots r(t-1) + R(t)$$

past reward

future reward

At each time step, a good strategy for an agent would be to **always choose an action that maximizes the (discounted) future reward.**

$$R(t)$$
$$= r(t) + \gamma r(t+1) + \gamma^2 r(t+2) + \cdots\cdots + \gamma^{n-t} r_{\mathrm{n}}(t)$$
$$= r(t) + \gamma R(t+1)$$

Introduction to Reinforcement learning

- Reinforcement learning
- Q-Learning

Q-Learning

- Q function represents the "quality" of a certain action in a given state.
- It is a table of states and actions.

$$Q(s(t), a(t)) = maxR(t+1)$$

Q-table

$Q[s, a]$	a_1	a_2	⋯	a_m
s_1				
s_2				
s_3				
⋮				
s_n				

$$\pi(s(t)) = \max_a Q(s(t), a)$$

choose an action that maximizes the future reward.

Q-Learning

- Bellman equation :

$$< s(t), a(t), r(t+1), s(t+1) >$$

$$Q(s(t), a(t)) = \max R(t+1)$$

$$Q\big(s(t), a(t)\big) = r(t+1) + \gamma \max R(t+2)$$

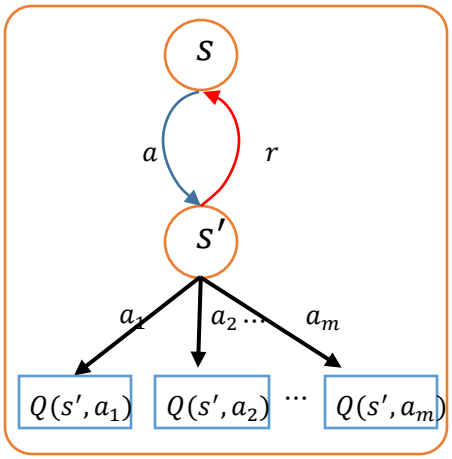
$$R(t+1) = r(t+1) + \gamma R(t+2)$$

$$Q\big(s(t), a(t)\big) = r(t+1) + \gamma \max_{a(t+1)} Q\big(s(t+1), a(t+1)\big)$$

$$Q(\mathrm{s}, a) = r + \gamma \max_{a'} Q(s', a')$$

current reward

maximum future reward from next state



Q-Learning

$$\begin{cases} < s, a, r, s' > \\ s \leftarrow s' \end{cases}$$

Q-table

$Q[s,a]$	a_1	a_2	$\cdots$	a_m
s_1				
s_2				
s_3				
$\vdots$				
s_n				

1. Algorithm Q-Learning
2. **Input:**
   1. S is a set of states
   2. A is a set of actions
   3. γ is the discount
3. initialize $Q[S, A]$ arbitrarily
4. observe initial state s
5. **Repeat:**
   1. select and carry out an action a, randomly
   2. receive reward r
   3. observe new state s'
   4. If s' is terminal state:
      1. $Q[s, a] = r$
   5. Else:
      1. $Q[s, a] = r + \gamma \max_{a'} Q[s', a']$
   6. $s \leftarrow s'$
6. **Until** terminated

Q-Learning

A tiny example:

Game description

States:

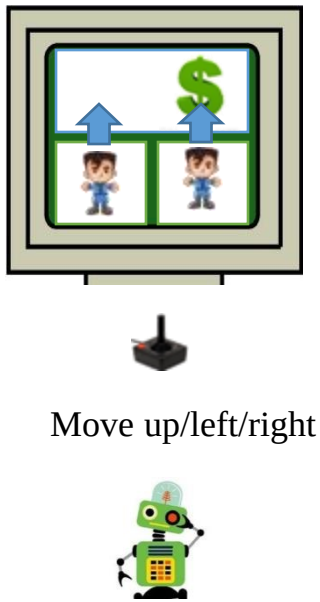
s_1, s_2, s_3, where s_3 is terminal state

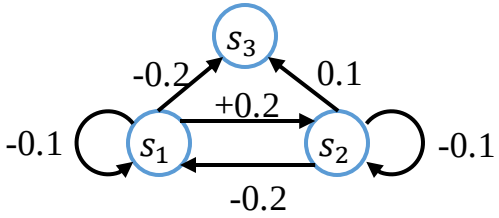
Actions:

a_1 denotes *up*. The agent goes up and moves to terminal state.

a_2 denotes *left*. The agent moves to left in state s_2 with a reward -0.2, while stay still in state s_1 with a reward -0.1.

a_3 denotes *right*. The agent moves to right in state s_1 with a reward 0.2, while stay still in state s_2 with a reward -0.1.





Q-Learning

Move up/left/right

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1			
s_2			
s_3	-	-	-

Algorithm Q-Learning

Input:

S is a set of states

A is a set of actions

γ is the discount

initialize $Q[S,A]$ arbitrarily

observe initial state s

Repeat:

select and carry out an action a, randomly

receive reward r

observe new state s'

If s' is terminal state:

$$Q[s,a] = r$$

Else:

$$Q[s,a] = r + \gamma \max_{a'} Q[s',a']$$

$s \leftarrow s'$

Until terminated

Q-Learning

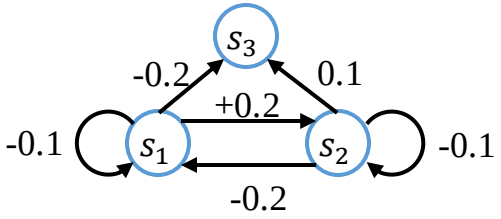
- Step 1: initialize $Q[S, A]$

  $\gamma = 0.8$

- Step 2:training loop

  1st episode:

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	0.60	0.74	0.94
s_2	0.36	0.32	0.78
s_3	-	-	-



$$s(0) = s_1, a(0) = a_3, r(1) = 0.2, s(1) = s_2 \;, a(1) = a_3, r(2) = -0.1, s(2) = s_2 \;, a(2) = a_1, r(3) = 0.1, s(3) = s_3$$

$$Q[s_1, a_3] = 0.2 + 0.8 * \max_{a_i}(Q[s_2, a_i]) = 0.2 + 0.8 * 0.78 = 0.82$$

$$Q[s_2, a_3] = -0.1 + 0.8 * \max_{a_i}(Q[s_2, a_i]) = -0.1 + 0.8 * 0.78 = 0.52$$

$$Q[s_2, a_1] = 0.1$$

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	0.60	0.74	0.82
s_2	0.36	0.32	0.78
s_3	-	-	-

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	0.60	0.74	0.82
s_2	0.36	0.32	0.52
s_3	-	-	-

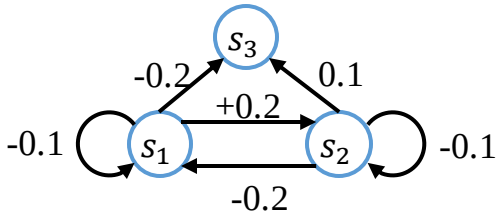
$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	0.60	0.74	0.82
s_2	0.1	0.32	0.52
s_3	-	-	-

$$Q(s,a) = r + \gamma \max_{a'} Q(s', a')$$

Q-Learning

After
11th episode

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.2	0.56	0.40
s_2	0.10	0.25	0.10
s_3	-	-	-



12st episode:

$s(0) = s_1, a(0) = a_2, r(1) = -0.1, s(1) = s_1 \quad , a(1) = a_1, r(2) = -0.2, s(2) = s_3$

$Q[s_1, a_2] = -0.1 + 0.8 * \max_{a_i}(Q[s_1, a_i])$
$= -0.1 + 0.8 * 0.56$
$= 0.35$

$Q[s_2, a_1] = -0.2$

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.35	0.40
s_2	0.10	0.25	0.10
s_3	-	-	-

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.35	0.40
s_2	0.10	0.25	0.10
s_3	-	-	-

$$Q(s,a) = r + \gamma \max_{a'} Q(s',a')$$

Q-Learning

After
15th episode

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.18	0.30
s_2	0.10	0.08	-0.00
s_3	-	-	-

After
50th episode

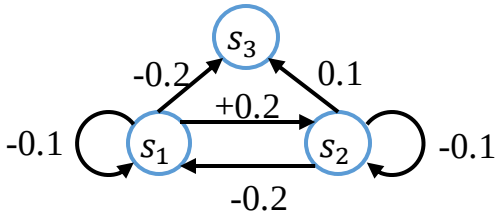
$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.12	0.28
s_2	0.10	0.02	-0.02
s_3	-	-	-

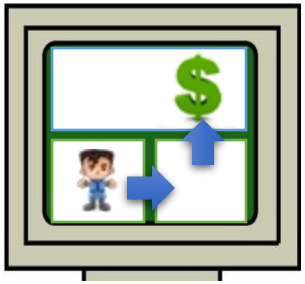
After
100th episode

$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.12	0.28
s_2	0.10	0.02	-0.02
s_3	-	-	-

After
1000th episode

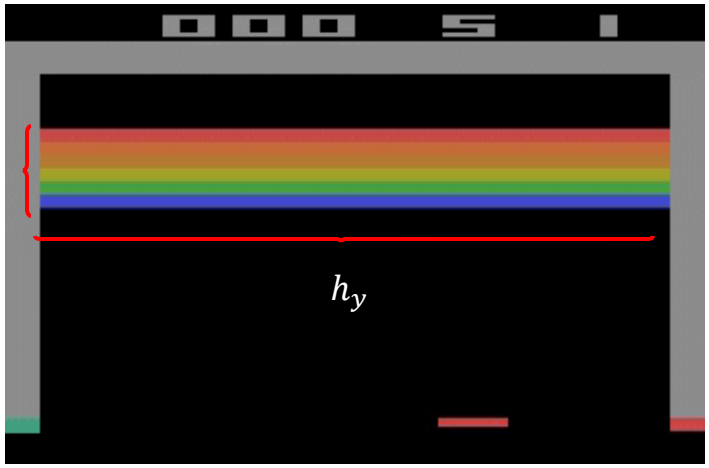
$Q[s,a]$	a_1 up	a_2 left	a_3 right
s_1	-0.20	0.12	0.28
s_2	0.10	0.02	-0.02
s_3	-	-	-





Move up/left/right

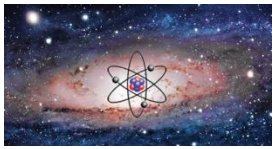
Q-Learning



Q-table

$Q[s,a]$	a_1	a_2	$\cdots$	a_m
s_1				
s_2				
s_3				
$\vdots$				
s_n				

$$\begin{cases} < s, a, r, s' > \\ s \leftarrow s' \end{cases}$$



$$< (3 * 256)^{w_x * h_y} < number\ of\ states$$

Too huge states space to approximate Q-function iteratively by Q-table!!!

Conclusion – Machine Learning

1. Supervised Learning
   - Linear Regression
   - Logistic Regression
   - Classification
     - Distance-based algorithms
     - Linear classifiers
     - Other classifiers
2. Unsupervised Learning
   - Clustering
     - K-means method
     - Spectral clustering
   - Representation learning
3. Reinforcement Learning
   - Q-Learning, Q-table
   - Exploration & Exploitation